A note on the use of the Puiseux Series in Celestial Mechanics

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Abstract. In this work a brief study of the singularities of the Hamilton-Jacobi equation is made in order to identify the different types of developments in series when a resonant occurs. The study is based in a application of the Weierstrass Preparation Theorem (Goursat, 1916) which defines cyclic system of roots and their corresponding Puiseux Series (Valiron, 1950; Dieudonné, 1971).

Keywords: Perturbation theory, resonant asteroids

1. Introduction

The first step in a application of a perturbation theory is the assumption that the functions involved may be expanded in power series of certain parameter, which is considered small. It is well known that if the problem is non resonant the series (even in the case of divergent series) are assumed in integer powers of this parameter. In the restricted three body problem, this parameter is the Jupiter mass. When a resonant occur, the expansions are not in integer power, but in square root (Poincaré, 1892) or in cubic root of the parameter (Ferraz-Mello, 1985a, 1985b). The question is how to determine which type of the development is necessary to use. The aim of this article is obtain the conditions for each type depending on model of resonance adopt.

2. Expansions about singular points of a algebraic equation

2.1. The Weierstrass Preparation Theorem

Theorem (Weierstrass): Let $F(x, \varepsilon)$ be a analytical function on the complex variables x, ε so that $F(0,0) = 0$. Let us consider that F may be expressed in the series:

$$F(x, \varepsilon) = A_0 + A_1(\varepsilon)x + ... + +A_n(\varepsilon)x^n + ... \qquad (1)$$

where each coefficient A_j is a power series in ε. If for $\varepsilon = 0$ the equation $F(x, 0) = 0$ has $x = 0$ as root of multiplicity k, then, $F(x, \varepsilon)$ may be *factorized as*

$$F(x,\varepsilon) = [x^k + \alpha_{k-1}(\varepsilon)x^{k-1} + ... + \alpha_1(\varepsilon)x + \alpha_0]G(x,\varepsilon) \qquad (2)$$

where the functions $\alpha_j(\varepsilon)$ *are analytical functions on* ε, $\alpha_j(0) = 0$ *and the function* G *satisfies* $G(0,0) \neq 0$.

The demonstration may be founded in Goursat (1916).

2.2. Critical Points. Puiseux Series

T search solutions in Puiseux series let us study the roots of the equation $F(x,0) = 0$. In a neighborhood of the origin in the complex plane, the roots of eqn. (2) are the roots of the equation $x^k + \alpha_{k-1}(\varepsilon)x^{k-1} + ... + \alpha_1(\varepsilon)x + \alpha_0 = 0$. Consider $\mathcal{D} \in C$ a domain which contain the origin (of the variable ε). For any value of $\varepsilon \in \mathcal{D}$, say, ε_0 we go to study the behavior of the roots of eqn. (2) by changing of ε_0, considered as a parameter.

Let $x_1^0, x_2^0, ..., x_k^0$ are the roots of the eqn.(2) for a certain ε_0. Consider the one-parameter transformation $\vartheta : C \times [0,k] \to C$ defined by:

$$(\varepsilon_0, l) \to \varepsilon_0 \exp(i2\pi l), \qquad 0 \le l \le k \qquad (3)$$

The eqn. (2) is transformed in

$$x^k + \alpha_{k-1}(\varepsilon_0 \exp(i2\pi l))x^{k-1} + ... + \alpha_1(\varepsilon_0 \exp(i2\pi l))x + \alpha_0(\varepsilon_0 \exp(i2\pi l)) = 0 \qquad (4)$$

Now, the upper-scripts 0 indicate the roots for $l = 0$. It is evident that for integer values of l the roots remain the same, our problem consist in consider the change in x by changing l continuously.

Let us introduce the following notation: Let x_r^m be the $r-th$ root of the equation (4) for $l = m$. A result of theory of functions is that for each r fixed, x_r^m is a continuous function in the parameter m.

Definition: Consider the equation $x^k + \alpha_{k-1}(\varepsilon)x^{k-1} + ... + \alpha_1(\varepsilon)x + \alpha_0 = 0$. *A subset of roots* $S_m = \{x_1^0, x_2^0, .., x_m^0,\} \in C$, $m \le k$ *form a cyclic system of order* m *if and only if* $x_j^m = x_j^0$. From this definition we can to state the following Lemma.

Lemma (Goursat): The roots of the equation $x^k + \alpha_{k-1}(\varepsilon)x^{k-1} + ... + \alpha_1(\varepsilon)x + \alpha_0 = 0$ *which are nulls for* $\varepsilon = 0$ *form one or more cyclic systems in the neighborhood of the origin.*

Each cyclic system has a unique Puiseux series: If we change $\varepsilon_0 = \varepsilon_0^{'m}$ after one loop about the origin (in the primed variable) return to the same value in ε_0. Since the root considered is in S_m the root after the loop return to the same value. This means that the each root as function on ε' is a unevaluated and has a development

$$x = \sum_{j=0}^{\infty} \lambda_j \varepsilon_0^{j'} \tag{5}$$

where x represent a element of S_m. If we return to ε_0, we obtain

$$x = \sum_{j=0}^{\infty} \lambda_j \varepsilon_0^{j/m} \tag{6}$$

which is the Puiseux series for each element of S_m.

To determine in a practical form how cyclic system are, we have the proposition.

Proposition: A polynomial $P(x) \in C[[x]]$ in the annulus of the polynomial with analytical coefficients has only one cyclic system if and only if it is irreducible

3. Application to the resonant-restricted three body Problem

3.1. Resonant Variables

In order to apply the precedent results we consider an asteroid moving around the Sun in a $(p+1):p$ mean-motion resonance whit Jupiter. Consider the resonant variables (Ferraz-Mello, 1987):

$$\begin{cases} \theta_1 = (p+1)\lambda_J - p\lambda - \varpi & J_1 = L - G \\ \theta_2 = (p+1)\lambda_J - p\lambda - \varpi_J & J_2 = G + \frac{1}{n_J}\Lambda \\ \theta_3 = \lambda - \lambda_J & J_3 = (p+1)L + \frac{p}{n_J}\Lambda \end{cases} \tag{7}$$

where λ, ϖ are the mean longitude and perihelion respectively. The subscript indicate Jupiter. n_J is the mean motion of Jupiter; L, G are the usual planar Delaunay variables and Λ is the momenta associated to the λ_J in the extended phase space.

3.2. Pendulum Model: Powers in square root in the Jupiter mass

After the first averaging over the fast angle θ_3 the Hamiltonian may be expanded about a reference value of J_3 which is a constant. If we choose $J_3 = L_{resonant}$. Since $L - J_3^* = -p(J_1^* + J_2^*)$ the expansion result:

$$\begin{aligned} H_0^* &= -\frac{\mu^2}{2(J_3^*)^2} - \frac{\mu^2}{2}\sum_{i=3}^{\infty}\frac{(i+1)}{J_3^{2+i}} p^i (J_1^* + J_2^*)^i \\ &- n_J J_3^* - g_J J_2^* - \varepsilon R \end{aligned} \tag{8}$$

For the application of the resonant perturbation theory (Ferraz-Mello, 1997) we separate the term that contains the slow angle θ_1 and we can write:

$$F_0^* = \frac{1}{2}\nu_{11}\xi^2 - \varepsilon R_1 \cos\theta_1 - \frac{\mu^2}{2}\sum_{i=3}^{\infty}\frac{(i+1)}{J_3^{2+i}}p^i\xi^i - \varepsilon\Delta R \tag{9}$$

where $\xi = J_1^* + J_2^*$

The Hamilton-Jacobi equation becomes in the algebraic equation

$$F_0^*(\xi,\varepsilon) = \frac{1}{2}\nu_{11}\xi^2 - \varepsilon R_1 \cos\theta_1 - \frac{\mu^2}{2}\sum_{i=3}^{\infty}\frac{(i+1)}{J_3^{2+i}}p^i\xi^i - \varepsilon\Delta R = 0, \tag{10}$$

which satisfies $F_0^*(\xi,\varepsilon) = 0$

Since $F(\xi, 0) = 0$ has $\xi = 0$ as root with multiplicity 2 and not depend on power greater than 2 in ε, we can write the equation

$$F(\xi,\varepsilon) = (\xi^2 + b\varepsilon)G(\xi,\varepsilon) \tag{11}$$

Following Valiron (1951) and Dieudonné (1971) we can write

$$\sqrt{\varepsilon/c} = \xi(1+\psi(\xi)), \tag{12}$$

and in a neighborhood of the origin we can write:

$$\xi = \sqrt{\varepsilon/c} + \lambda_2\left(\sqrt{\varepsilon/c}\right)^2 + \ldots + \lambda_n\left(\sqrt{\varepsilon/c}\right)^n + \ldots \tag{13}$$

Which is equivalent to say $\xi = \mathcal{O}(\sqrt{\varepsilon})$. This result permit to search the generating functions as series of power of the square root of the Jupiter mass.

3.3. Andoyer Model: Powers in cubic root in the Jupiter mass

In some cases, when the eccentricity of the asteroid is small, we can to approximate the momentum $J_1 = Le^2/2$ or $e = \sqrt{2J_1/L}$ (Ferraz-Mello, 1987). With this approximation the main resonant part of the Hamiltonian may be written:

$$F_0 = a\xi^2 + \varepsilon\left(\sqrt{2J_1}\cos\theta_1 + \beta\cos\theta_2\right) + \text{higher orders} \tag{14}$$

If we use the Sessin's Integral $\mathcal{G}$ (for a complete description see Ferraz-Mello, 1987) the new variables are

$$\begin{aligned} \Theta &= \arctan(H/K) \\ \mathcal{J} &= \frac{1}{2}(K^2+H^2) \\ \mathcal{G} &= J_1 + J_2 - \mathcal{J} = \mathcal{O}(\mathcal{J}) \end{aligned} \tag{15}$$

where $K = \sqrt{2J_1}\cos\theta_1 + \beta e_J \cos\theta_2$, $H = \sqrt{2J_1}\sin\theta_1 + \beta e_J \sin\theta_2$. In this variables, the Hamiltonian may be written as

$$F_0 = \alpha\mathcal{G}\mathcal{J} + a\mathcal{J}^2 + \varepsilon\sqrt{2\mathcal{J}}\cos\Theta + \text{higher orders} \tag{16}$$

Let $x = \sqrt{2\mathcal{J}}$, the Hamilton-Jacobi equation take the form:

$$F = ax^2 + bx^4 + \varepsilon\mu x + \mathcal{O}(\varepsilon x^2) = 0 \tag{17}$$

μ contains the cosine and coefficient. Divided the equation by x we obtain $F = ax + bx^3 + \varepsilon\mu + \mathcal{O}(\varepsilon x) = 0$. Remember that $a = \mathcal{O}(\S^{\in})$ and we can note that $F(0,0) = 0$. The equation $F(x,0) = 0$ has $x = 0$ as root with multiplicity 3. By application of the Weierstrass Preparation Theorem we can put:

$$F = (x^3 + \alpha_2(\varepsilon)x^2 + \alpha_1(\varepsilon)x + \alpha_0(\varepsilon))G(x,\varepsilon) \tag{18}$$

Rest to proof that the polynomial $x^3 + \alpha_2(\varepsilon)x^2 + \alpha_1(\varepsilon)x + \alpha_0(\varepsilon)$ is irreducible.

To proof this, we can use that $a = \mathcal{O}(x^2)$ and substitute in the equation, obtain:

$$F = ax^3 + \varepsilon\mu + \mathcal{O}(\varepsilon x) = 0 \tag{19}$$

In a neighborhood of the origin we can write:

$$\left(\frac{\varepsilon}{a'}\right)^{1/3} = x(1 + \phi(x)) \tag{20}$$

and in analogous way follow Valiron (1951) or Dieudonné (1971) we obtain the Puiseux series for the second Fundamental Model of resonance (Henrard & Lemaitre 1982; Ferraz-Mello, 1985a,1985b)

$$x = \left(\frac{\varepsilon}{a'}\right)^{1/3} + \lambda_2\left(\frac{\varepsilon}{a'}\right)^{2/3} + \ldots \tag{21}$$

Return to relation $x = \sqrt{2\mathcal{J}}$ we obtain that $\mathcal{J} = \mathcal{O}(\varepsilon^{2/3})$ which permit to assume the generating functions expanded in powers of $\varepsilon^{2/3}$.

4. Conclusion

This paper presents a method to obtain the differents developments in series when a resonance occurs. It is based in the application of the Weierstrass Preparation Theorem which permits to determine the type of development may be used depending on the integrable model describe the resonance.

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