Analytical Proper Elements for Hilda Asteroids

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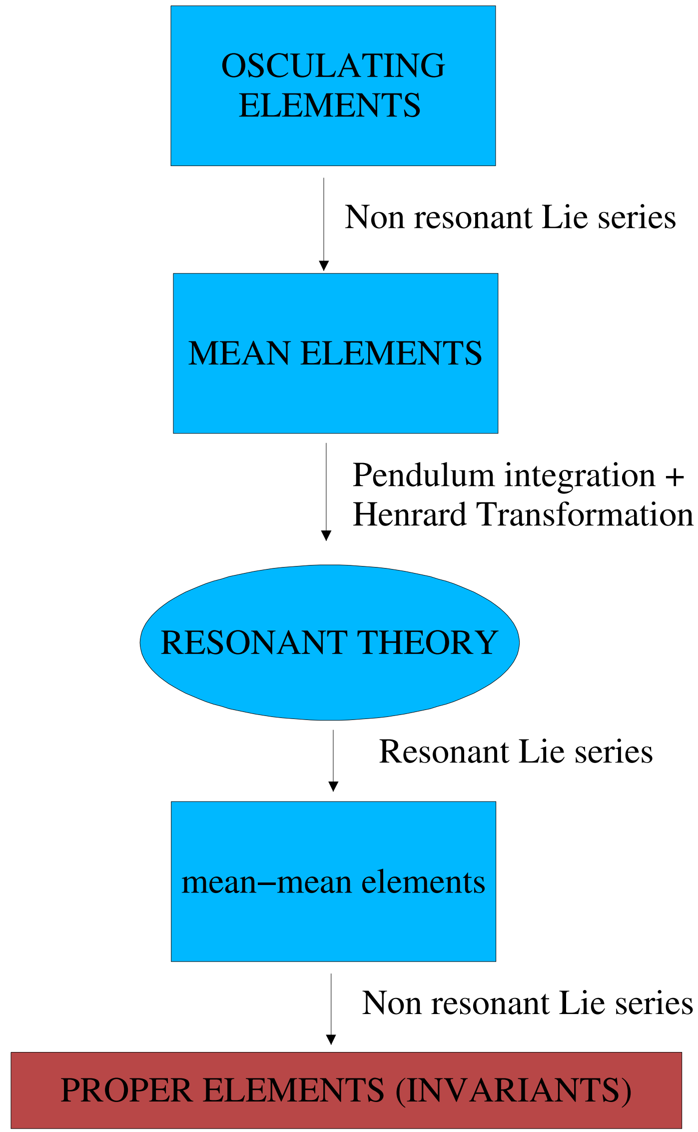
The scheme


OSCULATING ELEMENTS
Non resonant Lie series
MEAN ELEMENTS
Pendulum integration +
Henrard Transformation
RESONANT THEORY
Resonant Lie series
mean−mean elements
Non resonant Lie series
PROPER ELEMENTS (INVARIANTS)

The Resonant Coordinates

For the resonance 3:2 we have the standard resonant coordinates:

$$\begin{aligned}
\theta_1 &= 3\lambda_J - 2\lambda - \varpi \\
\theta_2 &= 3\lambda_J - 2\lambda - \varpi_J \\
\theta_3 &= \lambda - \lambda_J
\end{aligned}$$

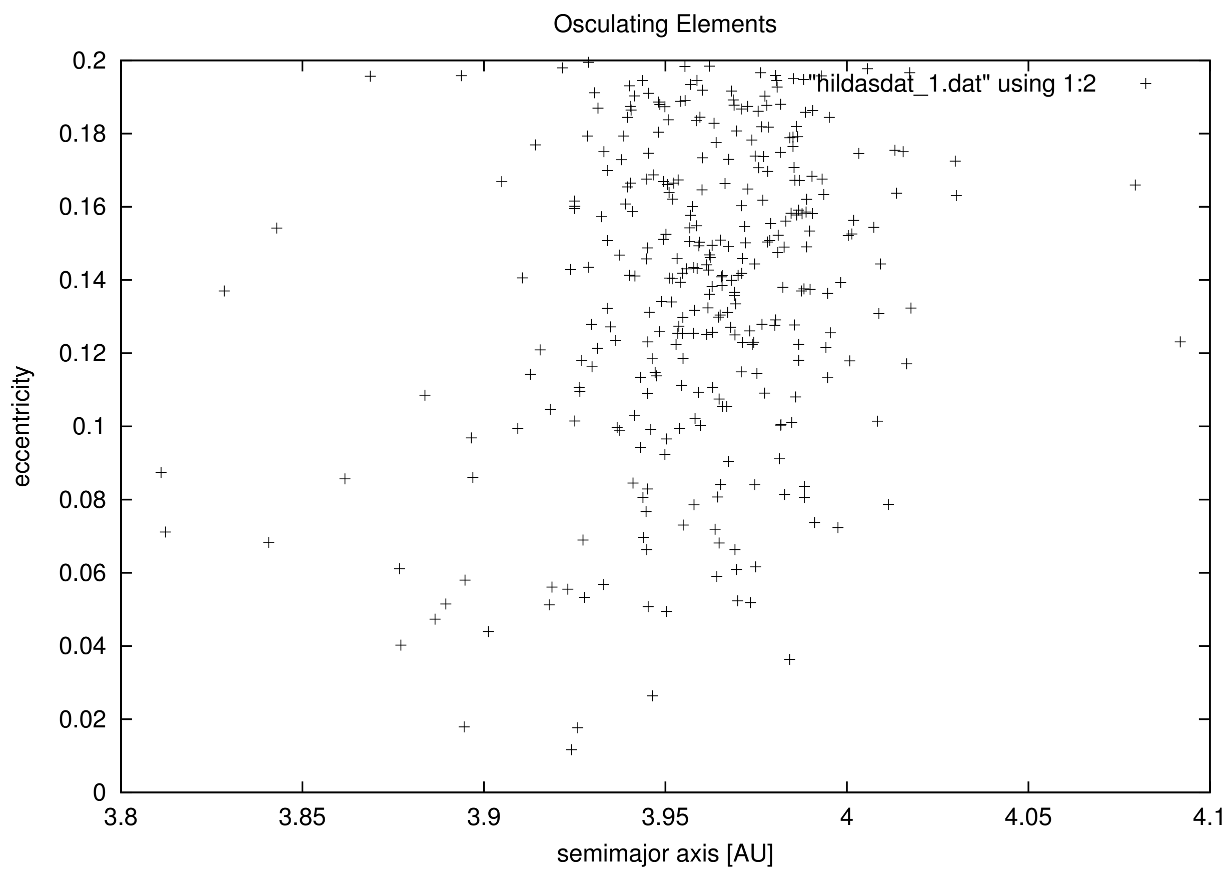
$$\begin{aligned}
J_1 &= L - G \\
J_2 &= \frac{n_J G + \mathcal{T}}{n_J - g_J} \\
J_3 &= 3L + 2\frac{(g_J G + \mathcal{T})}{n_J - g_J}
\end{aligned}$$

where, λ, ϖ are the mean longitude and perihelium respectively. L, G are the Delaunay moments (for the particle) and $\mathcal{T}$ is the momentum conjugated to the time.

The Hamiltonian Evolution

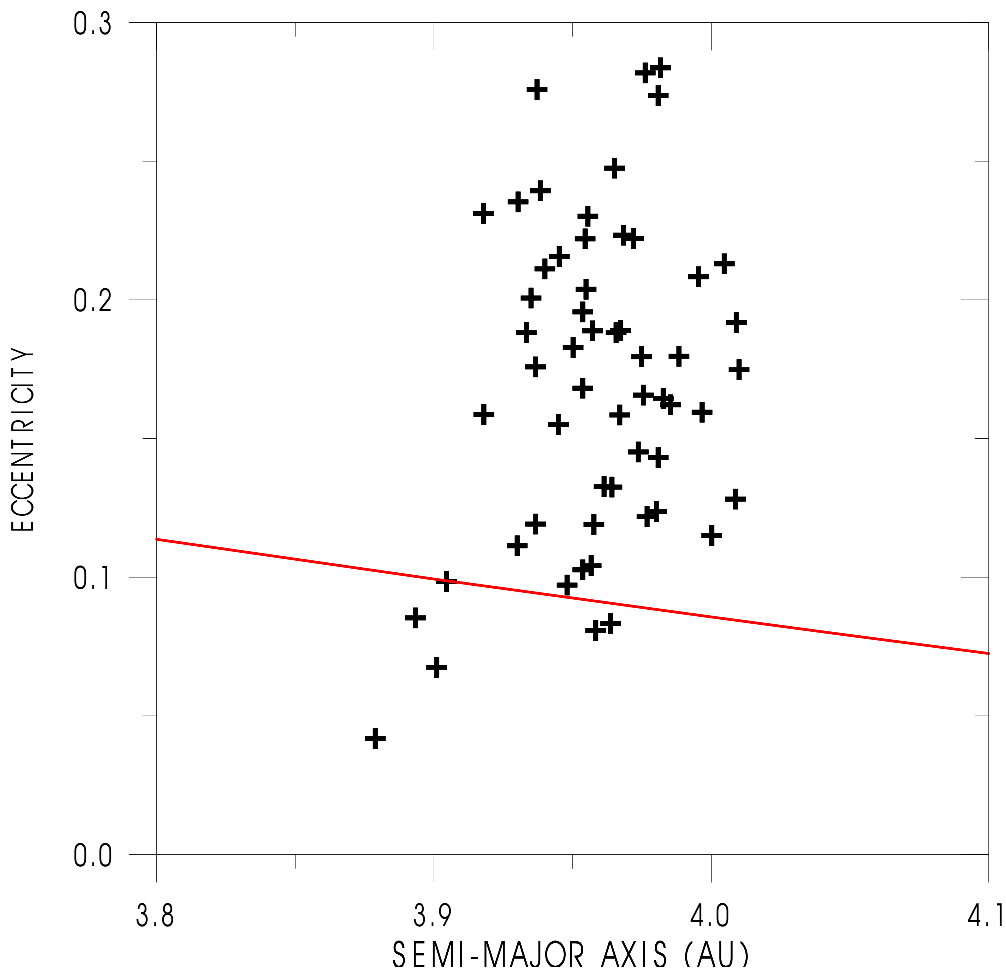
- Osculating Elements $H = H(\theta, J)$
  • Application of the standard Lie Series –>
- Mean Elements $H^* = H^*(\theta^*, J^*)$
- Resonant Theory –>
  • Formal integration for the simple pendulum (Hori Kernel)–>
  • The transformation is extended to include the second degree of freedom with the Henrard-Lemaitre formula.
  –> $H^* = H^*(w_1^*, w_2^*, \Lambda_1^*, \Lambda_2^*)$
  • Averaging over w_1 to get the *mean-mean Elements*
  $H^{**} = H^{**}(w_2^{**}, \Lambda_1^{**}, \Lambda_2^{**})$ –>
- Proper Elements
  $H^{***} = H^{***}(\Lambda_1^{***}, \Lambda_2^{***})$ –>
- Equivalent Orbital Elements a^{***}, e^{***} –>

Osculating Elements



Convergence of the Laplacian Expansion

RED LINE: Sundman radius of convergence:



The Disturbing Function

The expansion for the disturbing function is the *Beaugé's Development* (Beaugé, 1996) adapted to include the short-period terms

$$
\begin{aligned}
R &= \frac{1}{a_J} \sum_{i=0}^{4} \sum_{j=0}^{15} \sum_{k=0}^{15} \sum_{l=-15}^{15} \sum_{m=-15}^{15} \sum_{n=-15}^{15} R_{ijklmn} \times \\
&\quad (\alpha - \alpha_{res})^i e^j e_J^k \cos(l\theta_1 + m\theta_2 + n\theta_3) \\
e &= \sqrt{1 - \left[1 - \frac{J_1}{J_3 - 2(J_1 + J_2)}\right]^2}, \\
a &= \left[J_3 - 2(J_1 + J_2)\right]^2 / \mu,
\end{aligned}
$$

where R_{ijklmn} are constant coefficients, $\alpha = \frac{a}{a_J}$ and $\alpha_{res} = \frac{2}{3}^{\frac{2}{3}}$. where R_{ijklmn} are constant coefficients, $\alpha = \frac{a}{a_J}$ and $\alpha_{res} = \frac{2}{3}^{\frac{2}{3}}$, e is the eccentricity.

The Mean Elements

In the extended phase space (to eliminate the time dependence) the Hamiltonian is:

$$H = -\frac{\mu^2}{2L^2} + n_J \Lambda + \varepsilon R$$

let us consider the transformation:

$$H^* = E_{W^T} H(\theta_1^*, \theta_2^*, \theta_3^*; J_1^*, J_2^*, J_3^*)$$

$$W^T = W_1^T + W_2^T + \ldots$$

The perturbation equations are:

$$\begin{aligned} H_0^* &= \left(-\frac{\mu^2}{2L^2} + n_J \Lambda\right)_{(J_1^*, J_2^*, J_3^*)} \\ H_1^* &= \left(R + \left\{H_0^*, W_1^T\right\}\right)_{(\theta_1^*, \theta_2^*, \theta_3^*; J_1^*, J_2^*, J_3^*)} \end{aligned}$$

They are solved by the averaging rule

$$H_1^* =< R >_{\theta_3^*}$$

Preliminaries. Expansion of the Hamiltonian about a reference value

Once made the first average, the Hamiltonian does not depend on the angle θ_3 and we can choose

$$J_3^* = L_{res} = [\frac{2\mu^2}{3n_J}]^{1/3}$$

It is easy to show that, at the *exact resonance*

$$(J_1 + J_2) = 0$$

and, as $L - J_3^* = -2(J_1^* + J_2^*)$, we can expand the Hamiltonian in Taylor series around $\xi = (J_1^* + J_2^*) = 0$

The Expansions

For the unperturbed part (H_0) we have the series:

$$H_0 = -\frac{\mu^2}{2}\sum_{i=0}^{\infty}\frac{(i+1)}{J_3^{2+i}}2^i(J_1^* + J_2^*)^i + 3n_J(J_1^* + J_2^*) - n_J J_3 - g_J J_2$$

Each coefficient can be expanded in powers of $(J_1^* + J_2^*)$

$$\begin{aligned}
R_0 &= R_0^{(0)} + R_0^{(1)}(J_1 + J_2) + R_0^{(2)}(J_1 + J_2)^2 + \ldots \\
R_1 &= R_1^{(0)} + R_1^{(1)}(J_1 + J_2) + R_1^{(2)}(J_1 + J_2)^2 + \ldots \\
R_2 &= R_2^{(0)} + R_2^{(1)}(J_1 + J_2) + R_2^{(2)}(J_1 + J_2)^2 + \ldots
\end{aligned}$$

The Resonant Theory

The Hamiltonian $H^*(\theta^*, J^*)$ is a two-degree-of-freedom Hamiltonian where θ_1^* is critical (resonant) and θ_2^* is a long period angle.

- Split the Hamiltonian into two parts
$$H^*(\theta^*, J^*) = H^*_{pendulum} + \Delta H^*$$
- Integrate the Pendulum in the set $(\theta_1^*, (J_1^* + J_2^*))$ and obtain the angle-action variables (w_1^*, Λ_1^*)
- Extend the transformation to include the other degree of freedom via *Henrard-Lemaitre Transformation*
- Average over the fast angle w_1

The Hori Kernel

The main resonant part of the Hamiltonian is:

$$F_2^* = \frac{1}{2}\nu_{11}(J_1^* + J_2^*)^2 + \varepsilon\mathcal{R}_{00}^0(J_2^*) - g_J J_2^* + \varepsilon\mathcal{R}_{10}^0(J_2^*)\cos\theta_1^*$$

where $\nu_{11} = -\frac{12\mu^2}{(J_3^*)^4}$ and the functions $\mathcal{R}_{00}^0$ and $\mathcal{R}_{10}^0$ are generically defined by

$$\mathcal{R}_{\ell m}^0 = -\left(\sum_{i=0}^{4}\sum_{j=0}^{15}\sum_{k=0}^{15} R_{ijk\ell m0}\frac{(\alpha^* - \alpha_0)^i}{a_J} e^{*j} e_J^k\right)_{\xi=0}$$

Solution up to order $\mathcal{O}(\mathcal{Q}^7)$

The angle-action variables, in the case of small amplitud regime are defined by the relations:

$$
\begin{aligned}
J_1^* + J_2^* &= -8\frac{\omega_1^0}{|\nu_{11}|}\{[\cos w_1^*]\mathcal{Q} - [2\cos w_1^* - \cos 3w_1^*]\mathcal{Q}^3 \\
&- [\frac{17}{2}\cos w_1^* - 5\cos 3w_1^* - \cos 5w_1^*]\mathcal{Q}^5\} + \mathcal{O}(\mathcal{Q}^7)
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta_1^* &= 8\{[\sin w_1^*]\mathcal{Q} - [6\sin w_1^* - 3\sin 3w_1^*]\mathcal{Q}^3 \\
&- [\frac{25}{2}\sin w_1^* - 3\sin 3w_1^* - 5\sin 5w_1^*]\mathcal{Q}^5 + \mathcal{O}(\mathcal{Q}^7)\}
\end{aligned}
$$

where $\mathcal{Q}$ is the amplitude of oscillation of the pendulum:

$$\mathcal{Q} = \sqrt{\frac{\nu_{11}\Lambda_1^*}{32\omega_1^0}} \qquad \text{(small)}$$

The Henrard-Lemaitre Formula

To extend the canonical transformation to include the other degree of freedom we use the *Henrard-Lemaitre Formula*

$$\begin{aligned} \theta_2 &= w_2 - \Xi_2 \\ J_2 &= \Lambda_2 \end{aligned}$$

where:

$$\Xi_2 = \int_0^{w_1} \left(\frac{\partial \theta_1}{\partial w_1} \frac{\partial J_1}{\partial \Lambda_2} - \frac{\partial \theta_1}{\partial \Lambda_2} \frac{\partial J_1}{\partial w_1} \right) dw_1$$

The Hamiltonian up to order $\mathcal{O}(\varepsilon^2 Q^4)$

Up to order $\mathcal{O}(\varepsilon^2)$, the Hamiltonian can be written in the form:

$$H^* = F_2^*(\Lambda_1^*, \Lambda_2^*) + F_3^*(w^*, \Lambda^*)$$

where the subscripts mean the degree in $\sqrt{\varepsilon}$

$$F_2^* = \mathcal{F}(\Lambda_1^*, J_2^*) - g_J J_2^* + \varepsilon \mathcal{R}_{10}|_{\xi=0}(J_2^*) + \varepsilon \mathcal{R}_{00}^0(J_2^*),$$

$$\begin{aligned} F_3^* &= \frac{1}{6}\nu_{111}(J_1^* + J_2^*)^3 + \varepsilon(J_1^* + J_2^*)\frac{d\mathcal{R}_{00}}{d\xi}|_{\xi=0} + \\ &+ \varepsilon(J_1^* + J_2^*)\frac{d\mathcal{R}_{10}}{d\xi}|_{\xi=0}\cos\theta_1^* + \varepsilon\mathcal{R}_{11}^0\cos\theta_2^* + \\ &+ \varepsilon\mathcal{R}_{0-1}^0\cos(\theta_1^* - \theta_2^*). \end{aligned}$$

Cosines up to order $\mathcal{O}(\mathcal{Q}^4)$

$$\begin{aligned}
\cos\theta_1^* &= 1 + 16[-1 + \cos 2w_1^*]\mathcal{Q}^2 + 32[-\cos 2w_1^* + \cos 4w_1^*]\mathcal{Q}^4 + \\
&+ [-64 - 112\cos 2w_1^* + 128\cos 4w_1^* + 48\cos 6w_1^*]\mathcal{Q}^6 + \cdots .
\end{aligned} \tag{1}$$

Up to terms of order $\mathcal{Q}^3$, we also have:

$$\begin{aligned}
\cos\theta_2^* &= \cos w_2^* + 4[\cos(w_1^* + w_2^*) - \cos(w_2^* - w_1^*)]\mathcal{Q} + \\
&+ 8\{-2\cos w_2^* + (1 + \frac{1}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*})\cos(w_2^* + 2w_1^*) + \\
&+ (1 - \frac{1}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*})\cos(w_2^* - 2w_1^*)\}\mathcal{Q}^2 + \\
&+ \{(-\frac{32}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} - 24)\cos(w_1^* + w_2^*) + \\
&+ (-\frac{32}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} + 24)\cos(w_2^* - w_1^*) + \\
&+ (\frac{32}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} + 12)\cos(3w_1^* + w_2^*) + \\
&+ (\frac{32}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} - 12)\cos(w_2^* - 3w_1^*)\}\mathcal{Q}^3
\end{aligned} \tag{2}$$

cosines

$$\begin{aligned}
\cos(\theta_1^* - \theta_2^*) &= \cos w_2^* + \{-\frac{8}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\cos(w_2^* - 2w_1^*) + \\
&+ \frac{8}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\cos(2w_1^* + w_2^*)\}\mathcal{Q}^2 + \mathcal{O}(\mathcal{Q}^4)
\end{aligned} \tag{3}$$

Expression for F_3^*

$$
\begin{aligned}
F_3^* &= \Big[-\frac{8\varepsilon\omega_1^0}{|\nu_{11}|}\left(\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} + \frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\mathcal{Q} - \\
&+ \left(\frac{64\nu_{111}(\omega_1^0)^3}{|\nu_{11}|^3} - \frac{\varepsilon\omega_1^0}{|\nu_{11}|}\left(80\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} + 16\frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\right)\mathcal{Q}^3\Big] \cos w_1^* + \\
&- \left[\left(\frac{64\nu_{111}(\omega_1^0)^3}{3|\nu_{11}|^3} + \frac{72\varepsilon\omega_1^0}{|\nu_{11}|}\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} - \frac{8\varepsilon\omega_1^0}{|\nu_{11}|}\frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\mathcal{Q}^3\right] \cos 3w_1^* + \\
&+ \left[\varepsilon\mathcal{R}_{11}^0 + \varepsilon\mathcal{R}_{0-1}^0 - 16\varepsilon\mathcal{R}_{11}^0\mathcal{Q}^2\right] \cos w_2^* + \\
&- \left[4\varepsilon\mathcal{R}_{11}^0\mathcal{Q} - \left(24\varepsilon\mathcal{R}_{11}^0 - \frac{32\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^3\right] \cos(w_1^* - w_2^*) + \\
&+ \left[4\varepsilon\mathcal{R}_{11}^0\mathcal{Q} - \left(24\varepsilon\mathcal{R}_{11}^0 + \frac{32\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^3\right] \cos(w_1^* + w_2^*) + \\
&+ \left[\left(\frac{8\varepsilon\mathcal{R}_{0-1}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} + 8\varepsilon\mathcal{R}_{11}^0 + \frac{8\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^2\right] \cos(2w_1^* + w_2^*) + \\
&- \left[\left(\frac{8\varepsilon\mathcal{R}_{0-1}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} - 8\varepsilon\mathcal{R}_{11}^0 + \frac{8\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^2\right] \cos(2w_1^* - w_2^*) +
\end{aligned}
$$

and...

$$
\begin{aligned}
&+ \quad \left[\left(\frac{32\varepsilon\mathcal{R}^0_{11}}{|\nu_{11}|} \frac{\partial\omega^0_1}{\partial\Lambda^*_2} - 12\varepsilon\mathcal{R}^0_{11} \right) \mathcal{Q}^3 \right] \cos(3w^*_1 - w^*_2) + \\
&+ \quad \left[\left(\frac{32\varepsilon\mathcal{R}^0_{11}}{|\nu_{11}|} \frac{\partial\omega^0_1}{\partial\Lambda^*_2} + 12\varepsilon\mathcal{R}^0_{11} \right) \mathcal{Q}^3 \right] \cos(3w^*_1 + w^*_2) + \\
&+ \quad \mathcal{O}(\mathcal{Q}^4)
\end{aligned}
$$

The Homological Equation and the *mean-mean* Hamiltonian

$$\{F_2^*, W_2^*\}_1 = H_3^* - F_3(w^*, \Lambda^*)$$

The averaging rule

$$H_3^{**} = \frac{1}{2\pi}\int_0^{2\pi} F_3^* dw_1^{**}$$

The Hamiltonian for the *mean-mean elements* is:

$$\begin{aligned} H^{**} &= \mathcal{F}(\Lambda^{**}) - g_J\Lambda_2^{**} + \varepsilon\mathcal{R}_{10}|_{\xi=0}(\Lambda_2^{**}) + \varepsilon\mathcal{R}_{00}^0(\Lambda_2^{**}) + \\ &+ \left[\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{**}) + \varepsilon\mathcal{R}_{0-1}^0(\Lambda_2^{**}) - 16\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{**})\mathcal{Q}^2\right]\cos w_2^{**} \end{aligned} \quad (4)$$

The Resonant Lie Generator

The solution for the first homological equation is:

$$
\begin{aligned}
\omega_1 W_2^{(2)} \quad &= \quad \Big[-\frac{8\varepsilon\omega_1^0}{|\nu_{11}|}\left(\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} + \frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\mathcal{Q} - \\
&+ \quad \left(\frac{64\nu_{111}(\omega_1^0)^3}{|\nu_{11}|^3} - \frac{\varepsilon\omega_1^0}{|\nu_{11}|}\left(80\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} + 16\frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\right)\mathcal{Q}^3\Big] \sin w_1^{**} + \\
&- \quad \left[\left(\frac{64\nu_{111}(\omega_1^0)^3}{3|\nu_{11}|^3} + \frac{24\varepsilon\omega_1^0}{|\nu_{11}|}\frac{\partial\mathcal{R}_{10}}{\partial\xi}|_{\xi=0} - \frac{8\varepsilon\omega_1^0}{3|\nu_{11}|}\frac{\partial\mathcal{R}_{00}}{\partial\xi}|_{\xi=0}\right)\mathcal{Q}^3\right] \sin 3w_1^{**} + \\
&- \quad \left[4\varepsilon\mathcal{R}_{11}^0\mathcal{Q} - \left(24\varepsilon\mathcal{R}_{11}^0 - \frac{32\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^3\right] \sin(w_1^{**} - w_2^{**}) + \\
&+ \quad \left[4\varepsilon\mathcal{R}_{11}^0\mathcal{Q} - \left(24\varepsilon\mathcal{R}_{11}^0 + \frac{32\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^3\right] \sin(w_1^{**} + w_2^{**}) + \\
&+ \quad \left[\left(\frac{4\varepsilon\mathcal{R}_{0-1}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} + 4\varepsilon\mathcal{R}_{11}^0 + \frac{4\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^2\right] \sin(2w_1^{**} + w_2^{**}) + \\
&- \quad \left[\left(\frac{4\varepsilon\mathcal{R}_{0-1}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*} - 4\varepsilon\mathcal{R}_{11}^0 + \frac{4\varepsilon\mathcal{R}_{11}^0}{|\nu_{11}|}\frac{\partial\omega_1^0}{\partial\Lambda_2^*}\right)\mathcal{Q}^2\right] \sin(2w_1^{**} - w_2^{**}) +
\end{aligned}
$$

and....

$$
\begin{aligned}
+ & \left[\left(\frac{32\varepsilon\mathcal{R}^0_{11}}{3|\nu_{11}|}\frac{\partial\omega^0_1}{\partial\Lambda^*_2} - 4\varepsilon\mathcal{R}^0_{11}\right)\mathcal{Q}^3\right]\sin(3w_1^{**} - w_2^{**}) + \\
+ & \left[\left(\frac{32\varepsilon\mathcal{R}^0_{11}}{3|\nu_{11}|}\frac{\partial\omega^0_1}{\partial\Lambda^*_2} + 4\varepsilon\mathcal{R}^0_{11}\right)\mathcal{Q}^3\right]\sin(3w_1^{**} + w_2^{**}) + \\
+ & \mathcal{O}(\mathcal{Q}^4).
\end{aligned}
$$

The Last Integration

For commodity we use again a Lie series up to first order in the small parameter $\sqrt{\varepsilon}$ to integrate the Hamiltonian:

$$\begin{aligned} H^{**} &= \omega_1 \Lambda_1^{**} - g_J \Lambda_2^{**} + \varepsilon \mathcal{R}_{10}|_{\xi=0}(\Lambda_2^{**}) + \varepsilon \mathcal{R}_{00}^{0}(\Lambda_2^{**}) + \\ &+ \left[\varepsilon \mathcal{R}_{11}^{0}(\Lambda_2^{**}) + \varepsilon \mathcal{R}_{0-1}^{0}(\Lambda_2^{**}) - 16 \varepsilon \mathcal{R}_{11}^{0}(\Lambda_2^{**}) \mathcal{Q}^2\right] \cos w_2^{**} \end{aligned} \tag{5}$$

we define the frequency:

$$\omega_2 = \frac{\partial H_0^{**}}{\partial \Lambda_2^{**}}$$

Averaging

We adopt, again, the average:

$$H^{***} = \frac{1}{2\pi}\int_0^{2\pi} H^{**} dw_2^{**} \tag{6}$$

and the homological equation (obtained in the same way as in the first averaging) is:

$$\omega_2 \frac{\partial W_3^{(3)}}{\partial w_2^{***}} = \left[\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{***}) + \varepsilon\mathcal{R}_{0-1}^0(\Lambda_2^{***}) - 16\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{***})\mathcal{Q}^2\right]\cos w_2^{***}$$

whose solution is:

$$W_3^{(3)} = \frac{1}{\omega_2}\left[\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{***}) + \varepsilon\mathcal{R}_{0-1}^0(\Lambda_2^{***}) - 16\varepsilon\mathcal{R}_{11}^0(\Lambda_2^{***})\mathcal{Q}^2\right]\sin w_2^{***} \tag{7}$$

The Proper Elements

The resulting Hamiltonian is

$$H_0^{***} = \mathcal{F}(\Lambda^{***}) - g_J \Lambda_2^{***} + \varepsilon \mathcal{R}_{10}|_{\xi=0}(\Lambda_2^{***}) + \varepsilon \mathcal{R}_{00}^0(\Lambda_2^{***})$$

which defines two invariants Λ_1^{***} and Λ_2^{***} which are the *Dynamical Proper Elements*.

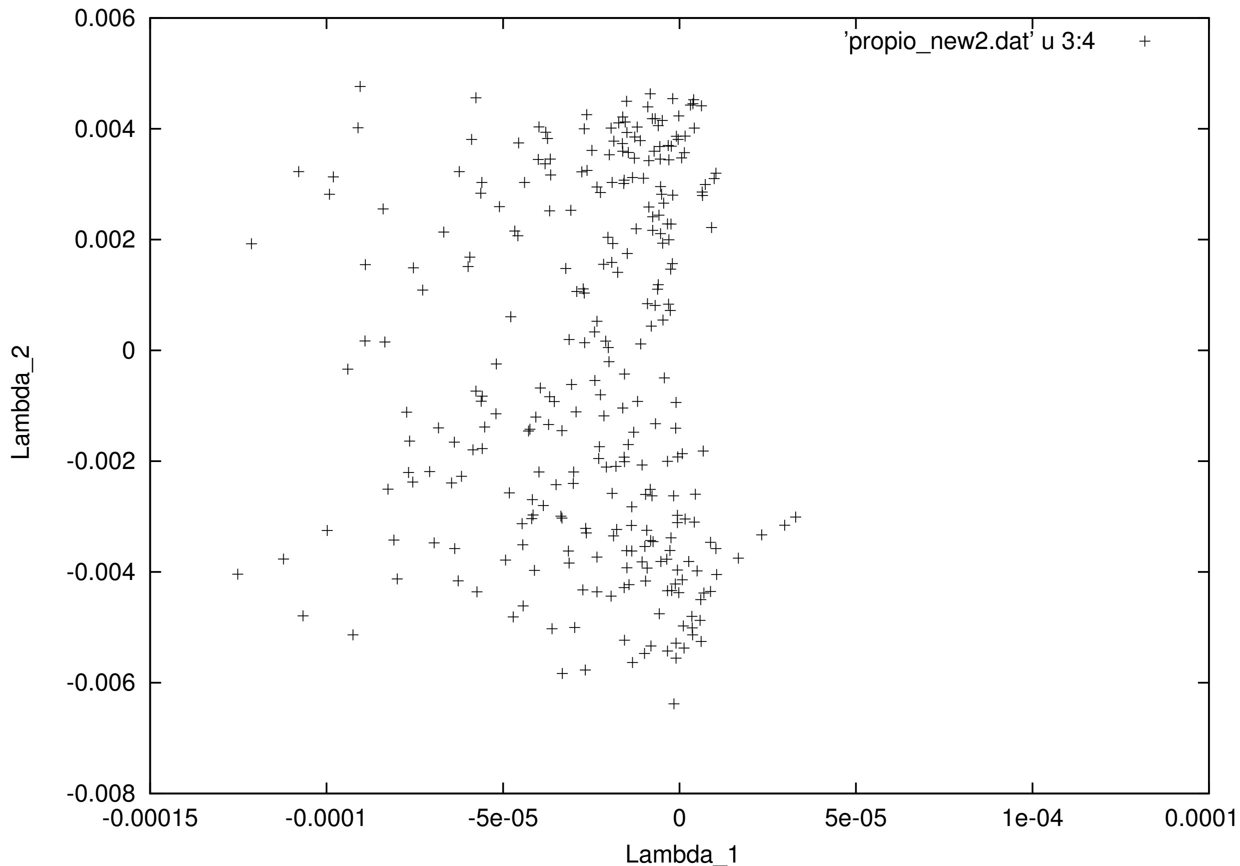
The Proper Elliptical Elements

It is interesting to write Λ_1^{***} and Λ_2^{***} in terms of the orbital elements.
The semi-major axis and eccentricity calculated from these elements will be called *equivalent elliptical elements* and they are calculated in the following way:

- With the invariants, calculate the formal J^{***} and θ^{***} variables
- Calculate the equivalent Delaunay's moments
- Obtain a^{***} and e^{***}

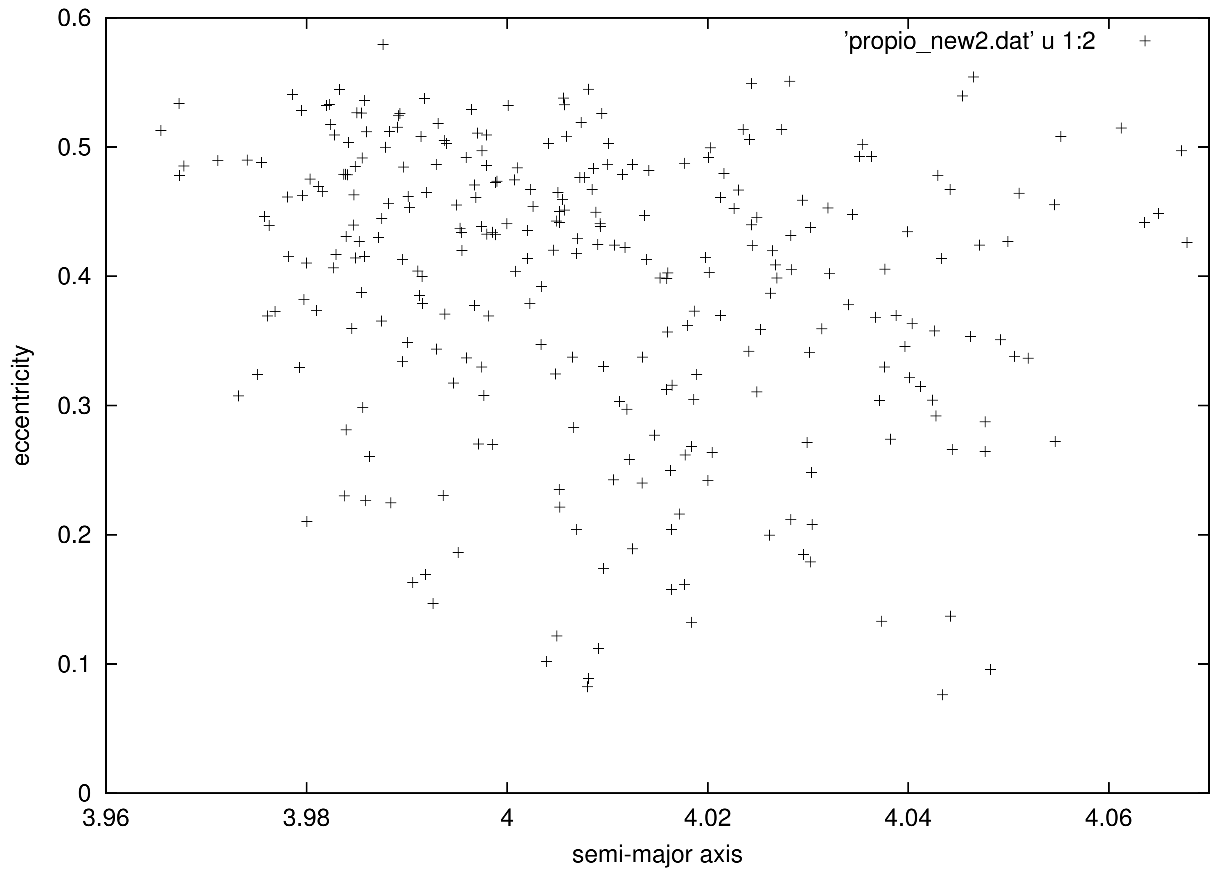
Preliminary Results

Dynamical Proper Elements



Preliminary Results

Elliptical Proper Elements



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